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ON THE DETERMINATION OF A SERIES OF STURM'S FUNCTIONS BY THE CALCULATION OF A SINGLE DETERMINANT.*

By E. B. Van Vleck.

1. In the most familiar form of Sturm's theorem, the number of roots of a real polynomial

$$f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n$$

included between two real limits, is determined with the aid of a series of functions consisting of the polynomial, its derivative, and the successive remainders, taken with appropriate signs, which occur in the process of finding the greatest common divisor of the polynomial and its derivative. In place of the derivative any other polynomial

$$f_1(x)=b_0x^n+b_1x^{n-1}+\cdots+b_n$$

may be employed as the divisor, if it has an odd number of roots between every two consecutive roots of $f(x)$. The resulting Sturm remainders, modified at pleasure by multiplication with a positive numerical factor, will here be denoted by $f_2, f_3, \ldots f_{n+1}$. The calculation of these remainders by the process of division is, in general, exceedingly laborious. The object of this paper is to show that the determination of the entire series of functions, with their proper signs, can be reduced to the systematic computation of a single determinant. The method suggested, it is believed, is well adapted to practical computation and, except in extremely simple cases, greatly abridges the labor.

To make clear the relation of the present article to previous work by others we shall bring together in §2 the various cases in which the coefficients of a series of Sturm's functions have all been expressed as minors of a common determinant.† We shall then go on to show, in §3, that a new case can be obtained by supplementing or completing certain results given incidentally in one of Sylvester's papers. Upon this new case will be based our method of computing the remainders.

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* This paper was read before the American Mathematical Society at the meeting of Feb. 25, 1899.

† These cases lie scattered in the mathematical literature, and do not seem to have been heretofore brought together.

2. The expression of the coefficients of a series of Sturm's remainders as minors of a common determinant was first effected by Jacobi* in 1835. Eighteen years later, in ignorance of Jacobi's work, the same result was also obtained by Sylvester.† Both writers express the coefficients as minors of Bezout's form of the resultant of f and f_1, namely

$$\begin{vmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ B_{n1} & B_{n2} & \cdots & B_{nn} \end{vmatrix}, \quad B_{ij} = B_{ji}$$

in which the constituents are certain well-known quadratic functions of the coefficients of f and f_1. The $n-i+1$ coefficients of the ith remainder are the minors obtained from the first i rows by associating those constituents which are found in the first $i-1$ columns with those of each succeeding column in turn.

The second series of Sturm's functions whose coefficients have been expressed in terms of a common determinant has a somewhat different composition.‡ If $f_1(x)$ is taken to be a polynomial of $n-1$th degree, the quotient f_1/f can be expanded, on the one hand, into a continued fraction

$$\frac{1}{a_1 x + b_1} - \frac{1}{a_2 x + b_2} - \frac{1}{a_3 x + b_3} - \cdots - \frac{1}{a_n x + b_n};$$

on the other hand, into an infinite series

$$\frac{c_0}{x} + \frac{c_1}{x^2} + \frac{c_2}{x^3} + \cdots\cdots$$

The denominators of the convergents of the continued fraction, inclusive of f itself, constitute a Sturm's series under the same restrictions for f_1 as in the case of Sturm's remainders. When obtained directly from the infinite series, the denominators have the form

$$\begin{vmatrix} c_0 & c_1 & c_2 & \cdots\cdots & c_{i-1} & 1 \\ c_1 & c_2 & c_3 & \cdots\cdots & c_i & x \\ c_2 & c_3 & c_4 & \cdots\cdots & c_{i+1} & x^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_i & c_{i+1} & \cdots & \cdots & c_{2i-1} & x^i \end{vmatrix},$$

* *Crelle*, Vol. 15.

† *Philosophical Transactions*, 1853. See, in particular, Sylvester's remarks on p. 433.

‡ Netto's *Algebra*, Siebente Vorlesung.

and their coefficients are therefore minors of the determinant

$$\begin{vmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_1 & c_2 & c_3 & \cdots & c_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_n & c_{n+1} & \cdots & \cdots & c_{2n} \end{vmatrix}.$$

In the special case in which f_1 is placed equal to the derivative of f, the constituent c_i is the sum of the ith powers of the roots of $f(x) = 0$, and the determinant, omitting the last row and column, becomes a familiar form of the discriminant of this equation.

The coefficients of the denominators of the same convergents have also been expressed* as minors of Sylvester's form of the resultant of f and f_1,

$$R = \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & \cdot\cdot & a_n & 0 & 0 & \cdot\cdot & 0 \\ b_0 & b_1 & b_2 & b_3 & \cdot\cdot & b_n & 0 & 0 & \cdot\cdot & 0 \\ 0 & a_0 & a_1 & a_2 & \cdot\cdot & a_{n-1} & a_n & 0 & \cdot\cdot & 0 \\ 0 & b_0 & b_1 & b_2 & \cdot\cdot & b_{n-1} & b_n & 0 & \cdot\cdot & 0 \\ 0 & 0 & a_0 & a_1 & \cdot\cdot & a_{n-2} & a_{n-1} & a_n & \cdot\cdot & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdot & b_0 & b_1 & b_2 & b_3 \cdot\cdot & b_n \end{vmatrix}, \qquad (1)$$

thus making a third case in which the coefficients of a Sturm's series have been expressed as minors of a common determinant. For the computation of the functions all these cases seem to be open to the common objection that the independent computation of the minors is far more laborious than the derivation of Sturm's remainders by the usual process of division. We shall therefore seek a series of functions whose coefficients form such a set of minors as permit of calculation simultaneously with the determinant.

3. To this end consider the set of polynomials which Sylvester† obtained by so applying his dialytic method of elimination — commonly used to obtain the above form of R — as to eliminate only the first i highest powers of x between $f(x) = 0$ and $f_1(x) = 0$, i taking successively the values 1, 2, , ... n. The coefficients of the ith polynomial thus obtained are the minors formed from the first $2i$ rows of (1) by associating those constituents which are contained in the first $2i-1$ columns with those of each

* Netto's *Algebra*, *l. c.*

† *Philosophical Transactions*, 1853, p. 426–7.

succeeding column in turn. Thus, for example, the second of these polynomials is

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ 0 & a_0 & a_1 & a_2 \\ 0 & b_0 & b_1 & b_2 \end{vmatrix} x^{n-2} + \begin{vmatrix} a_0 & a_1 & a_2 & a_4 \\ b_0 & b_1 & b_2 & b_4 \\ 0 & a_0 & a_1 & a_3 \\ 0 & b_0 & b_1 & b_3 \end{vmatrix} x^{n-3} + \cdots + \begin{vmatrix} a_0 & a_1 & a_2 & 0 \\ b_0 & b_1 & b_2 & 0 \\ 0 & a_0 & a_1 & a_n \\ 0 & b_0 & b_1 & b_n \end{vmatrix} = 0.$$

Sylvester has shown * that these polynomials are what he terms the "simplified residues" which occur in the process of finding the greatest common divisor of f and f_1, — that is, each remainder is so expressed that its coefficients are rational functions of the a and the b without a common divisor. In consequence each polynomial differs from the corresponding (simplified) Sturm remainder only by a numerical factor independent of the a and b. To demonstrate that these polynomials are *identical* with Sturm's remainders, it is left for us to prove the vital point that they agree in sign. It will evidently suffice to compare the leading coefficients of our polynomials with the corresponding coefficients of Sturm's remainders.

We will assume at the outset of the proof that the signs of a_0 and b_0 have been so taken as to be positive. This insures that the leading coefficient $\begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix}$ of our first polynomial shall agree in sign with the corresponding coefficient of the first of Sturm's remainders. Beginning with this polynomial and remainder, the degree of each succeeding polynomial, respectively remainder is, in general, one less than that of the preceding. Now Sylvester has remarked that when this is true of Sturm's remainders, and the leading coefficient of any remainder vanishes, the leading coefficients of the preceding and following remainders have opposite signs. This is evident; for let $L_0x^r + L_1x^{r-1} + L_2x^{r-2}$ and $M_0x^{r-1} + M_1x^{r-2} + M_2x^{r-3}$ denote the first three terms of two consecutive Sturm remainders. By division the first coefficient of the next simplified remainder is found to be

$$M_0(L_0M_2 - M_0L_2) - M_1(L_0M_1 - M_0L_1)$$

which for $M_0 = 0$ has a sign opposite to that of L_0. We will now prove that the same property also holds for our polynomials. It follows then at once that their leading coefficients agree in sign with the corresponding coefficients of Sturm's remainders.

* See the last reference. The same fact can also be inferred from an article by Netto in the *Festschrift der mathematischen Gesellschaft in Hamburg*, 1890, but the line of proof is not so simple or fundamental as Sylvester's.

Consider first the case in which $\begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix}$, the leading coefficient of the first polynomial, vanishes. The leading coefficient of the next polynomial is then equal to $-\begin{vmatrix} a_0 & a_2 \\ b_0 & b_2 \end{vmatrix}^2$ and consequently differs in sign from the first coefficient of $f_1(x)$.

Take next the leading coefficients of any three consecutive polynomials,

$$D_{2i-2} = \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & & \\ b_0 & b_1 & b_2 & \cdots & & \\ 0 & a_0 & a_1 & \cdots & & \\ \cdots & \cdots & \cdots & \cdots & & \\ 0 & 0 & 0 & \cdots & b_{i-1} & b_i \\ 0 & 0 & 0 & \cdots & a_{i-2} & a_{i-1} \\ 0 & 0 & 0 & \cdots & b_{i-2} & b_{i-1} \end{vmatrix}, \quad D_{2i} = \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & & \\ b_0 & b_1 & b_2 & \cdots & & \\ 0 & a_0 & a_1 & \cdots & & \\ \cdots & \cdots & \cdots & \cdots & & \\ 0 & 0 & 0 & \cdots & b_{i-1} & b_i \\ 0 & 0 & 0 & \cdots & a_{i-1} & a_i \\ 0 & 0 & 0 & \cdots & b_{i-1} & b_i \end{vmatrix},$$

$$D_{2i+2} = \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & & & \\ b_0 & b_1 & b_2 & \cdots & & & \\ 0 & a_0 & a_1 & \cdots & & & \\ \cdots & \cdots & \cdots & \cdots & & & \\ 0 & 0 & 0 & \cdots & b_{i-1} & b_i & b_{i+1} \\ 0 & 0 & 0 & \cdots & a_{i-1} & a_i & a_{i+1} \\ 0 & 0 & 0 & \cdots & b_{i-1} & b_i & b_{i+1} \end{vmatrix}$$

and multiply the first by the last with the aid of the following theorem of determinants:

"The product of a determinant and any one of its minors M is expressible as an aggregate of products of pairs of minors: the first factors of the products being obtained by taking q rows in which the rows of M are included and forming from them every minor of the qth order which contains M; the second factor of any product being that minor which includes M and the complementary of the first factor; and the sign of any product being fixed by transforming the second factor so as to have its principal diagonal coincident with those of the two minors which it was formed to include, and then taking + or − according as the sum of the numbers indicating the rows and columns from which the first factor was formed is even or odd." [Muir's *Theory of Determinants*, p. 128, § 90.]

In applying this proposition, the first of our three determinants can be taken out as a minor from the third in several ways. We will consider it to

be the minor which is obtained by omitting the first two and the last two rows, also the first column and the last three columns. We will choose for our q rows all the rows from the third to the last inclusive. Of the determinants of the $2i$th order which can be formed from these rows so as to contain D_{2i} as a minor, all but three include the first column which contains only zero constituents, and hence vanish identically. The remaining three are the determinants obtained by omitting, in addition to the first column, either the last column, the next to the last, or the second preceding the last. Of these three determinants the first is D_{2i}, while the third in accordance with the above theorem, is to be multiplied by D_{2i}. Hence when $D_{2i}=0$, the only partial product to be considered is the remaining determinant multiplied into a factor which is easily seen to be its negative. It follows that $D_{2i-2}\,D_{2i+2}$ is then negative. In other words, whenever the leading coefficient of any one of our polynomials vanishes, the leading coefficients of the preceding and following polynomials have opposite signs, as was to be proved. Since the same property holds for the remainders, and since, also, the first two polynomials and remainders have been shown to be identical, the subsequent polynomials must also be identical with the corresponding remainders.

The correctness of the signs of the leading coefficients of our polynomials might also have been inferred by reducing the resultant R to Bezout's form after the manner given in *Baltzer's Determinantentheorie*, § 11, section 14. Each principal minor* D_{2i} is at the same time transformed into the principal minor of Bezout's determinant which is of the ith order, that is, into the leading coefficient of one of the remainders.

4. We have thus demonstrated that f, f_1, *and the successive polynomials which are formed from the first* $2i$ *rows of* R, $(i=1, 2, \ldots n)$ *constitute a Sturm series.* A comparison of this series with the two series previously cited, whose coefficients were expressed as minors of a common determinant, shows certain advantages from a theoretical standpoint for either of the latter two. The order of the fundamental determinant is only one-half as great, and the determinant has the further advantage of being symmetrical with respect to the principal diagonal. But these advantages are largely offset by the fact that the constituents of the determinant (1) require no preliminary calculation. For example, if the number of real roots of $f(x)=0$ be desired, we need only to write down the series of principal minors of even order†

*German, *Hauptunterdeterminante.* By the term "principal minor" of the ith order is here to be understood the minor which contains the constituents common to the first i rows and columns.

†This series of determinants has been previously set up by Hurwitz, see *Math. Ann.*, Vol. 46.

$$a_0, b_0, \begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix}, \begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ 0 & a_0 & a_1 & a_2 \\ 0 & b_0 & b_1 & b_2 \end{vmatrix}, \ldots . R, \qquad (2)$$

these being the coefficients of the first terms, upon which the signs of the successive functions depend when we place $x = \pm \infty$. *The number of changes of sign in (2) gives the number of pairs of imaginary roots of the polynomial.* To determine how many of the real roots are positive, we must also take into consideration the constant terms of the polynomials, for which we have the expressions

$$a_n, b_n, \begin{vmatrix} a_0 & a_n \\ b_0 & b_n \end{vmatrix}, \begin{vmatrix} a_0 & a_1 & a_2 & 0 \\ b_0 & b_1 & b_2 & 0 \\ 0 & a_0 & a_1 & a_n \\ 0 & b_0 & b_1 & b_n \end{vmatrix}, \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & 0 \\ b_0 & b_1 & b_2 & b_3 & b_4 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_n \\ 0 & 0 & b_0 & b_1 & b_2 & b_n \end{vmatrix}, \ldots R. \qquad (3)$$

The excess of the number of changes of sign in (2) over the number of changes of sign in (3) is equal to the required number of positive roots. The series (2) has also a significance when $f_1(x)$ is not subject to the restriction stated in §1, giving then the value of Cauchy's index for the quotient f_1/f.

5. As soon as we leave abstract theory and pass to calculation, our series of functions has a most decided advantage, in that the significant constituents of any two consecutive rows of the fundamental determinant (1) are the same as the constituents of the two preceding rows. If therefore in any row the values of the constituents are changed by the addition of a multiple of the preceding row, exactly the same change can be made in the constituents of each alternate row thereafter without altering the value of any minor which appears as a coefficient in one of our Sturm's functions. Moreover the alternate rows can each be multiplied by the same positive number without affecting either the form of the determinant or the signs of these minors. If then, in this manner, beginning with the second row of the determinant and proceeding downward, we reduce to zero all constituents which lie to the left of the principal diagonal, the repetitive character of the determinant will always be preserved below the last row in which this reduction has been effected, while at the same time the number of zero constituents rapidly increases. At each stage of the reduction it will therefore be necessary to retain only two rows, the one in which the reduction is about to be made and the row above it with the aid of which the

reduction is effected. As soon as the first constituent of the former row has been reduced to zero, the latter row, with its constituents moved one place to the right, is to be written down below it as the next row to be modified, and so on. The determinant can thus rapidly be brought to the form

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & \dots & a_n & 0 & 0 \dots 0 \\ 0 & C_{20} & C_{21} & C_{22} & \dots & C_{2,n-1} & 0 & 0 \dots 0 \\ 0 & 0 & C_{30} & C_{31} & \dots & C_{3,n-2} & C_{3,n-1} & 0 \dots 0 \\ 0 & 0 & 0 & C_{40} & \dots & C_{4,n-3} & C_{4,n-2} & 0 \dots 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \dots C_{2n,0} \end{vmatrix} . \qquad (4)$$

The even rows furnish the coefficients of the successive Sturm remainders, the ith *remainder* f_{i+1} *being either* $C_{2i,0}\,x^{n-i} + C_{2i,1}\,x^{n-i-1} + \dots + C_{2i,n-i}$ *or the same taken with the opposite sign, according as there is an even or an odd number of negative constituents in the principal diagonal above the constituent* $C_{2i,0}$.

The advantage of this mode of computing a Sturm's series over the usual long-division method can perhaps be best seen by the reader, if he makes the computation for some equation by both methods. Take, for instance, the equation*

$$x^6 + x^5 - x^4 - x^3 + x^2 - x + 1 = 0,$$

and place $f_1(x)$ equal to the derivative of the left hand member. The two functions f_1 and f_2 are in consequence identical, and f_3 becomes the first remainder. Omitting the first row and column of (1), we have as our fundamental determinant

$$\begin{vmatrix} 6 & 5 & -4 & -3 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 6 & 5 & -4 & -3 & 2 & -1 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 6 & 5 & -4 & -3 & 2 & -1 \end{vmatrix} ,$$

* This equation is found in Serret's *Algebra*, Part I, chap. 6. Owing to an error committed in computing the first remainder, the series of Sturm's functions there given is incorrect. The error does not however affect the conclusion that the roots of the equation are all imaginary.

which reduced to its final form becomes

$$\left|\begin{array}{cccccccccc}
6 & 5 & -4 & -3 & 2 & -1 & & & & \\
 & 1 & -2 & -3 & 4 & -5 & 6 & & & \\
 & & 17 & 14 & -27 & 32 & -37 & & & \\
 & & & -8 & -4 & 6 & -8 & 17 & & \\
 & & & & 44 & -114 & 120 & -7 & & \\
 & & & & & -16 & 18 & -6 & 11 & \\
 & & & & & & -86 & 138 & 31 & \\
 & & & & & & & -30 & -46 & 43 \\
 & & & & & & & & 506 & -173 \\
 & & & & & & & & & -331 & 253 \\
 & & & & & & & & & & 1
\end{array}\right| .$$

According to the usual method the third Sturm remainder $-86x^2+138x+31$ is obtained by dividing $17x^4+14x^3-27x^2+32x-37$ by $-44x^3+114x^2-120x+7$, fractional coefficients being avoided by multiplying the minuends by appropriate positive numbers. Several of the coefficients occurring in this division exceed 100,000, the largest being 306,480, and after the division the factor 1734 must be removed from the remainder to reduce it to the above form. On the other hand, if the determinant be made the basis of the computation, to obtain the third remainder after the second has been found we have to reduce the matrix

$$\left|\begin{array}{ccccc}
44 & -144 & 120 & -7 & 0 \\
-8 & -4 & 6 & -8 & 17 \\
0 & 44 & -114 & 120 & -7
\end{array}\right|$$

in the manner already described. The largest number occurring in this reduction is 480 and the divisors removed from the second and third rows are 17 and 3. In the computation of the next remainder the ratio of the largest numbers involved in the two methods is approximately 60 : 1, and so on.

6. It will be noticed that the fundamental determinant (1) has the same general form whether read downwards and to the right from the upper left hand corner or read upwards and to the left from the lower right hand corner. If also, as in the example just given, $f_1(x)$ is a polynomial of the $n-1$th degree so that the first row and column may be erased, the determinant will be bordered both at top and bottom with the coefficients of the same polynomial. A second series of Sturm's functions can then be obtained by working upwards from the lowermost row and reducing all the constituents above the principal diagonal

to zero. The series of functions thus obtained* is identical with the series of remainders which Sturm derived by so conducting the division of $f(x)$ by $f_1(x)$ and each succeeding division thereafter, as to remove the two terms of lowest degree instead of the two terms of highest degree. After each division the factor $-x^2$ is to be removed from the remainder, which is then used as the next divisor.

7. Thus far we have not brought into consideration the functions

$$g_{i+1}(x) = C_{2i+1,0}\, x^{n-i} + C_{2i+1,1}\, x^{n-i-1} + \cdots$$

which are formed from the odd rows of (4) in the same manner as the functions $f_i(x)$ from the even rows. Between the functions f and g there exist relations having, in general,† the form

$$\begin{aligned}
&d_0 f_2 = -a_0 f_1 + b_0 f, \\
&d_1 g_2 = l_1 x f_2 + m_1 f, \qquad (5)\\
&\cdots\cdots\cdots\cdots \\
&d_{2i} f_{i+2} = l_{2i} g_{i+1} + m_{2i} f_{i+1}, \\
&d_{2i+1} g_{i+2} = l_{2i+1} x f_{i+2} + m_{2i+1} g_{i+1}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&d_{2n-2} f_{n+1} = l_{2n-2} x f_n + m_{2n-2} g_n ;
\end{aligned}$$

in which the l, m, and d denote constants whose values will presently be determined. For consider, say, the $(j+1)$th relation. The functions therein connected are formed from the jth, the $(j+1)$th, and the $(j+2)$th rows. Now immediately before the $(j+2)$th row was reduced to its final form, its significant constituents were the same as those of the jth row but displaced one column towards the right. The first of these constituents, lying to the left of the principal diagonal, was consequently to be reduced to zero. This can be effected by multiplying the row by $|C_{j+1,0}|$ and adding or subtracting the preceding row multiplied by $C_{j,0}$. Hence the three functions are connected in the manner stated. Furthermore if d_j denotes the greatest (positive) common divisor of the constituents of the $(j+2)$th row, which finally is removed to reduce the row to its simplest form, we have $l_j = \pm C_{j,0}$, $m_j = \pm C_{j+1,0}$. To remove the ambiguity of sign which here presents itself, suppose first that $C_{j+1,0}$ is positive. Then in the reduction of the $(j+2)$th row the preceding row, after multiplication with $C_{j,0}$, is to be subtracted. If also $C_{j,0}$ is positive, to obtain the coefficients of the three functions we either take the con-

* This series has been termed a "Sturm'sche Reihe zweiter Gattung." For a detailed discussion see Wendlandt's thesis bearing this title (Göttingen, 1877).

† The investigation following applies only to the general case.

stituents of the three rows without change of sign, or else change the sign of every constituent. It follows that $m_j = C_{j+1,0}$, $l_j = -C_{j,0}$. On the other hand, if $C_{j,0}$ is negative, to obtain the coefficients, we change either the signs of the constituents of the $(j+1)$th and $(j+2)$th rows or the signs of the constituents of the jth row. Hence l_j has the same value as before, but $m_j = -C_{j+1,0}$. The same results will also be obtained by similar reasoning when $C_{j+1,0}$ is negative. Hence l_j is in every case equal to $-C_{j,0}$, while m_j is equal to $+C_{j+1,0}$ or $-C_{j+1,0}$ according as $C_{j+1,0}$ is preceded in the principal diagonal by a positive or a negative constituent.

8. The relations (5) give at once a development of f/f_2 into a continued fraction

$$\lambda_1 + \frac{\mu_1}{\lambda_2} + \frac{\mu_2}{\lambda_3} + \cdots + \frac{\mu_{2n-1}}{1}, \quad \text{in which}$$

$$\lambda_{2i} = -\frac{l_{2i}}{m_{2i}}, \lambda_{2i+1} = -\frac{l_{2i+1}x}{m_{2i+1}}, \mu_i = \frac{d_i}{m_i}, \cdots \mu_{2n-1} = \frac{C_{2n-1,1}}{m_{2n-1}}.$$

From this it follows* that

$$f = \begin{vmatrix} \lambda_1-\mu_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & \lambda_2-\mu_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \lambda_3-\mu_3 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \lambda_{2n-1}-\mu_{2n-1} & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{vmatrix}.$$

A like expression for f_2 can obviously be obtained by omitting the first row and column and for f_{i+2} by omitting the first $2i+1$ rows and columns.

Consider now the case in which f_1 is of the $n-1$th degree so that we may set $f_2 = f_1$. *The problem of finding for any given numerical value of x the signs of all the successive remainders* — except the last which is a constant — *can evidently then be presented as that of determining the signs of the determinant last given and certain minors of odd order.*† For this purpose a slightly modified form of the determinant will be much more convenient. After the values of the λ and μ above given have been substituted, let each row be cleared of fractions by multiplying it into the modulus of the denominator

* Heine's *Kugelfunctionen*, Bd. I, s. 260–262.

† The use of the partial quotients of a continued fraction of a somewhat different form for the purpose of determining the signs of the remainders was suggested by Sylvester, *Philosophical Magazine*, 1853.

of its constituents. The composition of the determinant will then be the same as would result by constructing it directly from (4) according to the following rules:

(1). The principal diagonal is first to be filled with the same constituents as the principal diagonal of (4), the last constituent being replaced by unity, and every other constituent, beginning with the first, is to be multiplied by x.

(2). The parallel file just below the principal diagonal is to be filled with the constituents of the principal diagonal of (4), beginning with the third, but the sign of each constituent is to be reversed in case it is preceded in the principal diagonal of (4) by a negative constituent. The last constituent of the file is to be placed equal to unity.

(3). The parallel file just above the principal diagonal is to be filled with the negatives of the divisors d used in the reduction of the fundamental determinant. The last constituent of the file is to be placed equal to $-C_{2n-1,1}$.

(4). All other constituents of the determinant are zero.

(5). Finally, if any constituent of the file below the principal diagonal is negative, the signs of the three significant constituents of the row in which it is contained are to be reversed.

Thus, for example, if $f(x) = x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1$, the fundamental determinant and the divisors d used in its formation are as follows:

d										
	1	1	−4	−3	3	1				
		5	4	−12	−6	3				
1			1	−8	−9	12	5			
11				4	3	−6	−2			
5					−7	−6	10	4		
1						−3	−2	2		
4							−1	4	3	
7								−2	−1	
3									3	2
1										1

,

and the auxiliary determinant formed with the aid of the above rules is

$$\begin{vmatrix}
x & -1 & & & & & & & & \\
1 & 5 & -11 & & & & & & & \\
 & 4 & x & -5 & & & & & & \\
 & & 7 & -4 & 1 & & & & & \\
 & & & 3 & -7x & -4 & & & & \\
 & & & & 1 & -3 & -7 & & & \\
 & & & & & 2 & -x & -3 & & \\
 & & & & & & 3 & 2 & 1 & \\
 & & & & & & & 1 & 3x & -2 \\
 & & & & & & & & 1 & 1
\end{vmatrix}$$

The calculation of the minors whose signs are desired for the given value of x can be easily made by computing the minors common to the last i rows and columns, i taking successively the values 2, 3, 4, ... Each of these minors can be quickly computed since it is a linear combination of the two preceding, in which the coefficients depend only upon the three new constituents appearing in the minor. This method of determining the signs of Sturm's functions requires the substitution of the given value of x in a single determinant instead of in n functions, as by the usual method. In many cases, at least, the method can be employed with decided advantage.

MIDDLETOWN, CONN., MAY, 1899.

CONCERNING THE TRACTRIX OF A CURVE, WITH PLANIMETRIC APPLICATION.

By Derrick N. Lehmer.

1. We define the *l-Tractrix of a curve* as follows:

While a point (x,y) follows a given curve, a second point (ξ,η) moves directly toward it, or directly from it, keeping at a constant distance l from the first point. We will call the path of the second point an l-Tractrix of the given curve.

If $f\ (x,y)=0$ be the equation of the given curve, the equation of the l-Tractrix may be obtained by eliminating x and y from the three equations

$$\text{I} \qquad f\ (x,y)=0,$$

$$\text{II} \qquad (x-\xi)^2+(y-\eta)^2=l^2,$$

$$\text{III} \qquad (y-\eta)=\frac{d\eta}{d\xi}(x-\xi).$$

In fact, II gives the condition that the two points be at a constant distance apart, while III is the condition that the point (x,y) should lie on the tangent to the l-Tractrix at the point (ξ,η). The actual elimination in any special case may be a matter of great difficulty.

It is evident that a curve may have an infinite number of l-Tractrices depending on the choice of the initial point of the l-Tractrix. The l-Tractrix, on the other hand, has but one base curve, if the sign of l is taken into account. The base curve is, in fact, obtained by measuring a constant distant l from the point of tangency along each tangent to the Tractrix.

2. In what follows we shall have occasion to consider areas traced out by a point which describes a closed contour. The analytical formula for such an area is

$$\int_c \left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt$$

where x and y, — the coordinates of the moving point, — are functions of the independent variable t the time; the integral being taken around the curve so as to keep the area on the left. If the point moves in the opposite direction the area is to be taken *negatively*. It may easily happen that the positive and negative portions of an area may cancel, as in the Lemniscate of Bernoulli. If the contour is described n times the area is multiplied by n.

We shall also consider areas swept out by a revolving line. The sign of the area will depend on the direction of rotation of the line, being positive when in the direction of increasing angles.

3. THEOREM. *If a curve and its l-Tractrix are both closed, the area of the curve is equal to the area of the l-Tractrix plus k times the area of a circle of radius l, where k is the number of complete revolutions made by a tangent line in going around the l-Tractrix.*

We may replace condition number II of §1 by the two

$$x = \xi + l\cos\theta,$$
$$y = \eta + l\sin\theta,$$

where θ is the inclination to the axis of x of the line joining the two points. From these we get easily, taking the time t as independent variable,

$$(1)\qquad \left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = \left(\xi\frac{d\eta}{dt} - \eta\frac{d\xi}{dt}\right)$$
$$+ l\left(\xi\cos\theta\frac{d\theta}{dt} + \eta\sin\theta\frac{d\theta}{dt} - \sin\theta\frac{d\xi}{dt} + \cos\theta\frac{d\eta}{dt}\right) + l^2\frac{d\theta}{dt}.$$

Now it is easily verified that the second term on the right is equal to

$$l\left[\frac{d}{dt}(\xi\sin\theta - \eta\cos\theta) - 2\left(\frac{d\xi}{dt}\sin\theta - \frac{d\eta}{dt}\cos\theta\right)\right].$$

But from condition III of §1,

$$\frac{d\eta}{d\xi} = \frac{y-\eta}{x-\xi} = \frac{\sin\theta}{\cos\theta};$$

so that

$$\frac{d\eta}{dt}\cos\theta - \frac{d\xi}{dt}\sin\theta = 0,$$

and the equation (1) becomes

$$(2)\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = \left(\xi\frac{d\eta}{dt} - \eta\frac{d\xi}{dt}\right) + l\frac{d}{dt}(\xi\sin\theta - \eta\cos\theta) + l^2\frac{d\theta}{dt}.$$

We may choose our origin at the initial point (x,y) and let the axis of x coincide with the initial position of the line between (x,y) and (ξ,η). We have then the initial values:

$$\xi = l,$$
$$\eta = 0,$$
$$\theta = 0.$$

Now when the point (x,y) moves around the given curve, the integral

$$\int \left(x\frac{dy}{dt} - \eta\frac{dx}{dt} \right) dt$$

becomes equal to $2A$, where A is the area enclosed by the curve.

At the same time the integral

$$\int \left(\xi\frac{d\eta}{dt} - \eta\frac{d\xi}{dt} \right) dt$$

has taken the value $2T$, where T is the area of the l-Tractrix taken with proper sign. The final values of ξ and η are the same as the initial, while θ has changed to $2k\pi$. Also at both limits

$$\xi \sin\theta - \eta\cos\theta = 0,$$

and we have

$$2A = 2T + 2k\pi l^2,$$

or

$$A = T + k\pi l^2, \qquad \text{Q. E. D.}$$

4. As an example of the above theorem we may take for our tractrix a regular triangle. (See Fig. 1.)

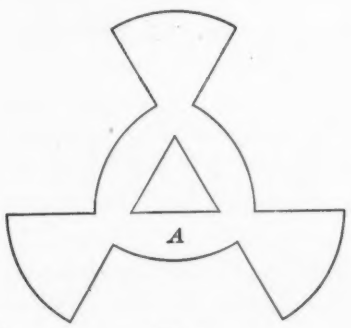

FIG. 1.

If we call Δ the area of the triangle, it is easily seen that

$$A = -2\Delta + \pi l^2.$$

Now in order that the outer curve be closed, it is seen to be necessary to traverse the contour of the l-Tractrix *twice*. It is further seen that the area of the l-Tractrix is traced *negatively*. Hence

$$T = -2\Delta$$

and so

$$A = T + \pi l^2.$$

5. The theorem of § 3 may be proved by the methods of infinitesimal geometry as follows:

Let a tangent line of fixed length l measured from the point of tangency, move along a curve. The area swept out is equal to $l^2\,\theta/2$, where θ is the difference in direction between the initial and final positions of the tangent line.

In fact the element of area is a triangle two sides of which are l and $l+ds$ while the included angle is $d\theta$. Dropping infinitesimals of the second order, the area is $l^2\,d\theta/2$, and integrating, the area is $l^2\,\theta/2$.

Otherwise, we may divide the curve into n equal parts and allow the tangent line to move along the broken line formed by joining the n points of division. The area generated is $l^2\,\theta'/2$ where θ' is the angle between the first and final chords. By increasing n indefinitely the theorem follows.

In this theorem the sign of the area is always to be considered. As an example we may take the well-known equitangential curve. The total change in the direction of the tangent line is $\pi/2$. Therefore the area bounded by the two axes and the curve is $\pi\,l^2/4$ which is a known theorem of the calculus.

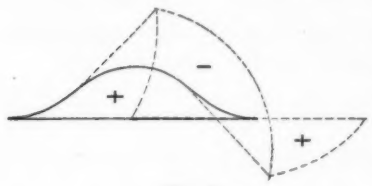

FIG. 2.

It may easily happen that the area swept out is zero as in fig. 2. Here $\theta = 0$ and the positive and negative areas exactly cancel.

Let now the curve be closed. If the tangent line makes k complete revolutions in going around it, the area generated will be $k\,\pi\,l^2$. Now this curve is the l-Tractrix of the curve generated by the extremity of the moving tangent. The areas of these two curves differ then by $k\,\pi\,l^2$ as in the theorem of § 3.

In a curve such as the cardioid it will be observed that k is not a whole number, but at the same time the curve generated by the extremity of the moving tangent is *not closed*, so that the theorem of § 3 does not apply. By tracing the curve twice however this difficulty is removed.

6. The theorem of § 3 may be made the basis of the theory of a planimeter of recent invention* which consists essentially of a bar of length l, carrying a tracing point at one end and a knife-edge at the other. (A pocket knife, with a blade at each end, opened up until the points are at a distance l apart, answers very well.) As the tracing point moves along a curve, the knife-edge traces out an l-Tractrix of that curve. The method of using the instrument is as follows: Start with the tracing point at the centre of gravity of the area to be measured. Proceed along some definite line to the contour.

* See Prytz, *Engineering*, Vol. 57, p. 813. Editorial, *Engineering*, Vol. 57, pp. 687, 725. Hill, *Phil. Mag.*, June 22, 1894.

Trace the contour and return along the same line to the starting point. Measure now the distance between the initial and final positions of the knife-edge. Multiply this distance by l. The result is the required area.

This extraordinary rule does not always give good results. The feature of starting at the centre of gravity of the area is non-essential as we shall see. Writers on the theory of the instrument have occupied themselves chiefly with the error introduced by starting at a point not exactly at the centre of gravity. Captain Prytz — who seems to be the inventor of the instrument — discusses it by means of infinite series, as does also Mr. Hill. The editorials cited in *Engineering* treat it by geometrical methods. In none of the discussions, however, can one get a definite idea of the magnitude and sign of the error which may arise in the use of the instrument, or how to better the approximation in any particular case.

7. The theorem of § 3 is not immediately applicable. The tractrix of a given closed curve is not necessarily itself a closed curve. We meet this difficulty as follows (see fig. 3):

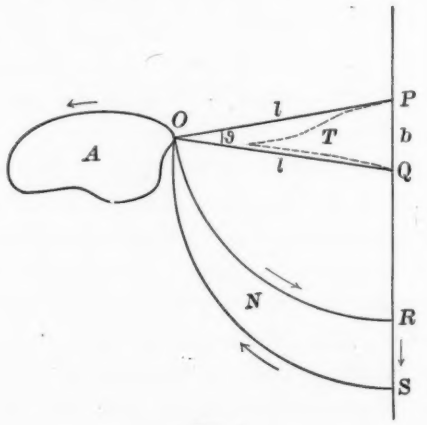

FIG. 3.

Suppose that while the tracing point starting from a point O on the boundary of A describes the contour of A in a positive sense, the knife-edge moves along the l-tractrix from Q to P. Join P and Q. With centre P and radius l describe the arc OR. With centre Q and radius l describe the arc OS. (R and S are on the line PQ.)

If the tracing point describes the contour $ORSO$ the knife-edge will return from P to R and the tractrix is now a closed curve. Our theorem then gives

$$A + N = T + k\pi l^2,$$

where all the areas are to be taken with their proper signs. Thus in the figure A is positive while N and T are negative, k being zero.

It is an easy matter to show that the area N is numerically equal to

$$\frac{l^2}{2}(\theta + \sin\theta),$$

where θ is the angle Q O P. We have then in general

$$A = T - \frac{l^2}{2}(\theta + \sin\theta) + k\pi l^2.$$

As an example we may take a circle of radius l and start with the tracing point at the centre O with the knife-edge at Q on the circumference.

Move now the tracing point from O to P to bring the knife-edge to the centre O. Describe the contour and return to O. The knife-edge moves back to Q, and the rule given in §6 would make the area zero. The above formula gives $A = \pi l^2$ since we have $\theta = 0$, $T = 0$, and $k = 1$.

8. We can make θ very small by increasing l. For small values of θ we may put $\sin\theta = \theta$, and the value $\frac{l^2}{2}(\theta + \sin\theta)$ is a little smaller than $l^2\theta$. But $l^2\theta = l \cdot l\theta$ which is a little larger than lb where b is the chord PQ. For $\theta \leqq 30°$ the error introduced by putting lb for $\frac{l^2}{2}(\theta + \sin\theta)$ is less than two parts in one thousand.

Also for values of l which are large compared with the longest line of the area to be measured, the line l will not turn a complete revolution. So k will be zero and we shall have

$$A = T - lb.$$

In starting from the centre of gravity it may happen that the area T is made up of positive and negative portions which very nearly cancel each other. In that case lb is a good approximation to the area A, and the rule given in § 6 is justified.

It is safer to use the formula

$$A = T - bl,$$

actually computing the area T. For large values of l the l-Tractrix is practically a rectilinear figure whose area is easily computed. In fact, by starting on the contour with the tracing point, instead of at the centre of gravity, we may often get a tractrix which differs very little from a triangle whose base is b. The height h may be measured and the area is

$$A = b(\tfrac{1}{2}h - l).$$

If the instrument were made to draw the tractrix one could draw the triangle and judge with considerable accuracy the magnitude and sign of his error.

Of course other errors enter, due to the drifting of the knife-edge; difficulty in tracing the contour; difficulty of measuring l and b, etc. With a little care, however, surprisingly accurate results may be obtained.

UNIVERSITY OF CHICAGO, MAY, 1899.

THE "NO-ROLLING" CURVES OF AMSLER'S PLANIMETER.

By F. Morley.

1. The Planimeter. I shall first restate the usual explanation of Amsler's instrument in kinematical language. According to my experience, kinematical intuitions should be freely used in elementary courses on Calculus.

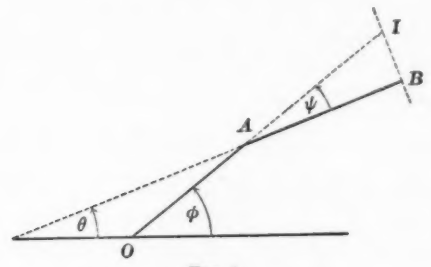

Fig. 1.

OA, AB (fig. 1) are rods of lengths a and b. They are hinged at A, and O is fixed. We suppose that B describes a circuit and that the rods oscillate back to their old position. And we assume that the area χ is known to be

$$\frac{1}{2}\int pv\, dt,$$

where pv is the moment of the velocity about O.

The velocity of B is made up of

$a\dot{\theta}$ perpendicular to OA,

$b\dot{\phi}$ " " OB.

The moments about O of these are

$$a\dot{\theta}(a+b\cos\psi) \text{ and } b\dot{\phi}(a\cos\psi+b).$$

The sum of these is the resultant moment. Hence

$$pv = a^2\dot{\theta}+b^2\dot{\phi}+ab(\dot{\theta}+\dot{\phi})\cos\psi.$$

Hence the area is

$$\chi = \frac{1}{2}\int[a^2\dot{\theta}+b^2\dot{\phi}+ab(\dot{\theta}+\dot{\phi})\cos\psi]\, dt;$$

or, since the change of θ and of ϕ is 0,

$$\chi = \frac{1}{2} ab \int (\dot{\theta} + \dot{\phi}) \cos \psi \, dt;$$

or, since $\dot{\psi} = \dot{\theta} - \dot{\phi}$,

$$\chi = \frac{1}{2} ab \int (2\dot{\theta} - \dot{\psi}) \cos \psi \, dt$$

$$= ab \int \dot{\theta} \cos \psi \, dt.$$

Now a wheel placed on AB at a distance c from A has, perpendicular to AB, a velocity

$$a\dot{\theta} \cos \psi + c\dot{\phi},$$

and therefore the record of the wheel is

$$\int (a\dot{\theta} \cos \psi + c\dot{\phi}) \, dt,$$

$$\text{or } a \int \dot{\theta} \cos \psi \, dt.$$

Hence the area described is

$$b \times \text{record of the wheel.}$$

2. The curves. In a second mode of explanation, indicated in Williamson's *Integral Calculus* and suggested by Sir Robert Ball, use is made of the curves for which the wheel slides without rolling. On examining these curves I find some facts which appear worth mentioning.

I suppose the wheel now to be at the point B, and the curves are those described when the velocity of B across AB is zero, or

$$a\dot{\theta} \cos \psi + b\dot{\phi} = 0.$$

These curves are the "orthogonal trajectories" of the curves for which B moves at right angles to AB, that is of the ring of circles of radius b whose centres lie on a circle of radius a. Thus, as a geometric definition, our curves are *the orthogonal curves of a ring of circles.* Or, in analogy with the orthogonal trajectory of a row of equal circles, we can call such a curve *the tractrix of a circle.*

If we invert the ring of circles as to its centre O, we have again a ring of

circles, and the old orthogonal curves pass into the new orthogonal curves. Thus the inverse of a tractrix as to O is again a tractrix.

In the case when $b = a$, the ring of circles inverts into the tangents of a circle, and the orthogonal curve of the tangents is the involute of the circle. Thus this special tractrix is *the inverse of the involute of a circle, as to the centre of the circle.*

3. Stereographic projection of the curves. To generalize the last remark, consider the two concentric circles which touch all the circles of the ring. Their radii are $r = a + b$, $r_1 = | a - b |$. Invert the plane into a sphere of diameter $\sqrt{r r_1}$, touching the plane at the origin O. The two concentric circles thus become equal and parallel circles of the sphere, of radius $\frac{\sqrt{r r_1}}{r + r_1}$. Call these *latitude circles.* The ring becomes a system of circles on the sphere, touching the latitude circles. When $b < a$, this system is a ring of small circles, whose planes all touch a right cylinder through the latitude circles, and no ground is gained. But when $b > a$, the system is a ring of great circles, whose planes all touch a right cone through the latitude circles.

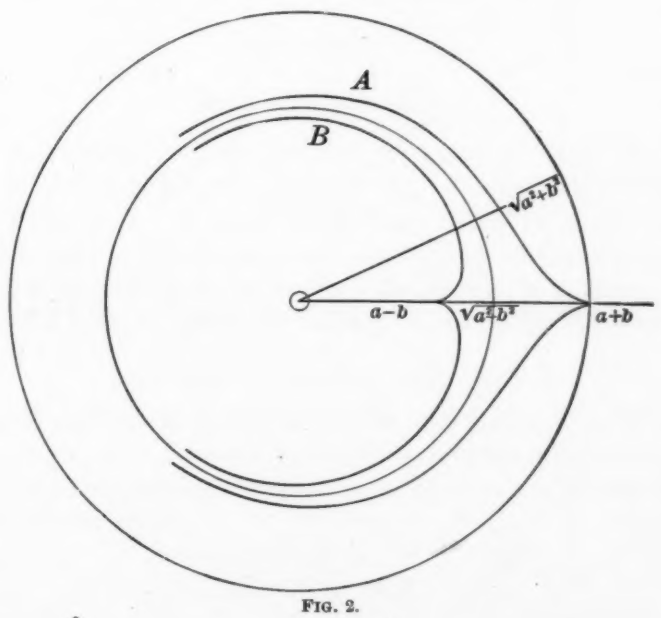

FIG. 2.

Take then the case $b > a$. The tractrix in the plane maps into the orthogonal curve of the tangent planes of a right cone, that is *into the spherical involute*

of a latitude circle, or curve traced by a point of a circle, whose centre is fixed, which rolls on a plane.

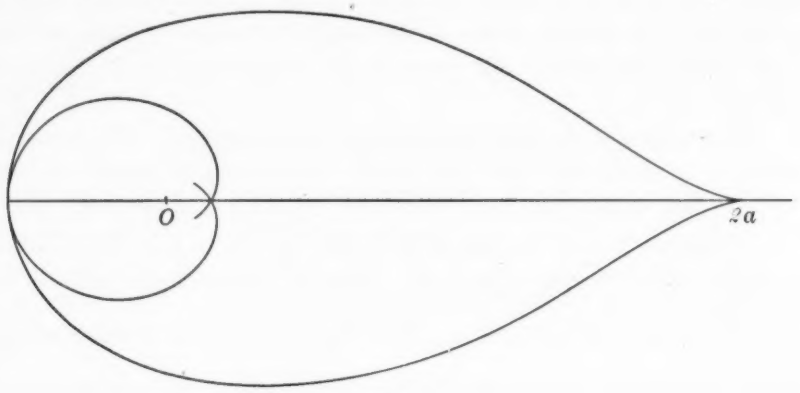

FIG. 3.

Since each point of the rolling circle is moving at right angles to its plane, the curve on the sphere is *a curve of constant slope*, that is the tangent makes always a constant angle with a given line.

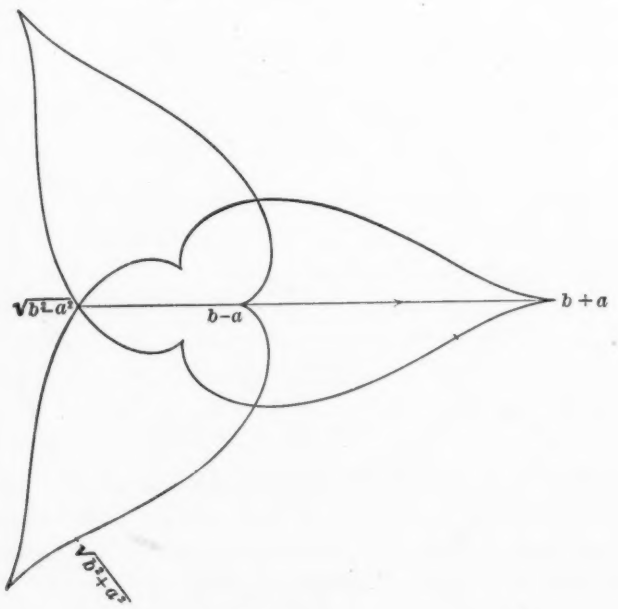

FIG. 4.

4. The form of the tractrix of a circle. The three cases, $b < a$, $b = a$, $b > a$ are illustrated in the figures. In (2) there is an asymptotic circle of

radius $\sqrt{a^2-b^2}$. In (3) the asymptotic circle reduces to a point. In (4), the case of most interest, we have equidistant cusps arranged along the circles $b \pm a$. The angular distance from cusp to cusp is determined at once from the stereographic projection (§3). For the great circle, whose perimeter is $\pi\sqrt{b^2-a^2}$, rolls on the latitude circles, whose radius is $(b^2-a^2)/2b$. Hence from one cusp to the next the angle is

$$\pi b/\sqrt{b^2-a^2},$$

and the curve is closed when $b/\sqrt{b^2-a^2}$ is a rational number.

5. The evolute. The instantaneous centre of the motion is I the intersection with OA of the perpendicular to AB through B. The centrode in the moving plane is the line BI. The motion is the rolling of this line on the fixed centrode, which is therefore the evolute of our curve. For the evolute, or locus of I, we have (fig. 1):

$$\text{radius vector } r = OI = c + b \sec \psi,$$
$$\text{perpendicular on tangent } p = b + a \cos \psi,$$

and therefore,

$$(r-a)(p-b) = ab,$$

or

$$a/r + b/p = 1.$$

Also

$$\text{arc } s = BI = b \tan \psi.$$

If now this evolute rolls on the line IB which is fixed, the curve itself will always pass through B, the point A will be fixed, and O will describe a circle whose centre is A. That is, the problem of finding a curve which, when rolled on a line, has as roulette a circle is solved by taking for the curve the evolute of the tractrix of a circle, and for the tracing point the centre of that circle. That this is in fact the only solution is seen at once from the formula:

$$\text{curvature of roulette} = D_p(p/r).$$

6. The Newtonian Catenary. As the evolute of the ordinary or right tractrix is the catenary, it is natural to ask under what force, tending to O or from O, will a string assume the form of the evolute of the tractrix of a circle. From the equations:

$$\text{tension } T \propto 1/p,$$
$$\text{force } F \propto D_r T,$$

we have in our case

$$F \propto D_r(1 - a/r) \propto 1/r^2.$$

Thus the requisite law is that of the inverse square.

Naturally, for attraction that part of the evolute must be taken which is convex to O, for repulsion the concave part; and there can be no inflexions at a finite distance.

The determination of the particular catenary, when the length of the string and the terminals are given, requires a closer study of the curve. We have (§5):

$$r = a + b \sec \psi, \tag{1}$$
$$p = b + a \cos \psi, \tag{2}$$
$$s = b \tan \psi, \tag{3}$$

and $rD_s\theta = \cos\psi$, in any curve, so that here

$$d\theta = \frac{b\, d\psi}{b + a\cos\psi},$$

$$\frac{\theta}{2b} = \frac{1}{\sqrt{b^2-a^2}} \tan^{-1}\left(\sqrt{\frac{b-a}{b+a}} \tan\frac{\psi}{2}\right),$$

or if $$\lambda = \sqrt{b^2-a^2}/b,$$

$$\tan(\lambda\theta/2) = \sqrt{\frac{b-a}{b+a}} \tan\frac{\psi}{2},$$

$$\cos\lambda\theta = \frac{a+b\cos\psi}{b+a\cos\psi},$$

and finally we have the polar equation

$$\frac{a^2-b^2}{r} = a - b\cos\lambda\theta. \tag{4}$$

Also $$\sin\lambda\theta = \frac{\sqrt{b^2-a^2}\sin\psi}{b+a\cos\psi}$$

$$= \lambda s/r,$$

or $$\lambda s = r\sin\lambda\theta. \tag{5}$$

The initial position has been that in which the point B is at its maximum distance from O; for the case when, in the initial position, the rods are doubled up, we change the sign of a. Whether the curves so obtained are to be regarded as distinct or as parts of one and the same curve is to some extent a matter of taste; but I take the latter view, which avoids distinguishing the cases when λ is rational, irrational, and imaginary.

The equations (4) and (5) serve to determine the position of the apse line and the values of a and b when the terminals, the length of string, and (in the case of repulsion) the number of times the curve is to enclose the centre, are given.

For given terminals and length of string we have in general *two* solutions, one for repulsion and one for attraction. The relation of these is one of considerable intricacy. But in the special case when the terminals are equidistant from the centre, the apse line is the same in both curves. Hence from (5) where s, r, θ are given, and s is less than r, *the two values of* λ *are the same.* Hence we may put

$$a_1 = \mu a, \qquad b_1 = \mu b$$

But from (1) and (3)

$$r^2 + 2ar = b^2 - a^2 + s^2,$$

and

$$r^2 - 2a_1 r = b_1^2 - a_1^2 + s^2,$$

or

$$r^2 - 2\mu a r = \mu^2 (b^2 - a^2) + s^2.$$

Therefore

$$(\mu + 1)(r^2 - s^2) = (\mu + \mu^2)(b^2 - a^2),$$

or

$$r^2 - s^2 = (b - a)\mu(b + a)$$

$$= \text{product of apsidal distances.}$$

That is, *when a string of length* $2s$ *is attached to two points at a distance* r *from the centre, and* $s < r$, *if* c *be the apsidal distance for attraction, and* c_1 *that for repulsion, then*

$$cc_1 = r^2 - s^2.$$

7. The form of the evolute. The general form of the curve should be noted; especially as the text-books on strings avoid giving figures.

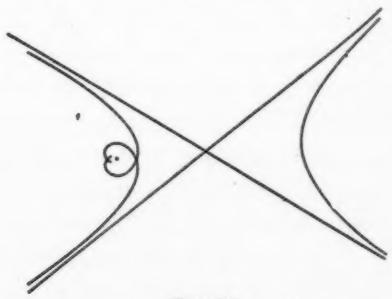

FIG. 5*A*.

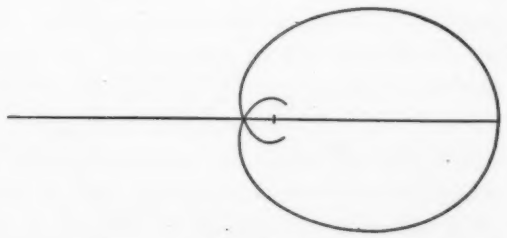

FIG. 5*B*.

Figs. 5 *A* and 5 *B* represent the evolutes of a curve of the form 2; and give a fair idea of the whole class. Fig. 7 is the curve (a circular quartic)

for which $\lambda = \frac{1}{2}$; the general case can be grasped from this by adding more loops round the centre, and by rotating the whole about the centre through

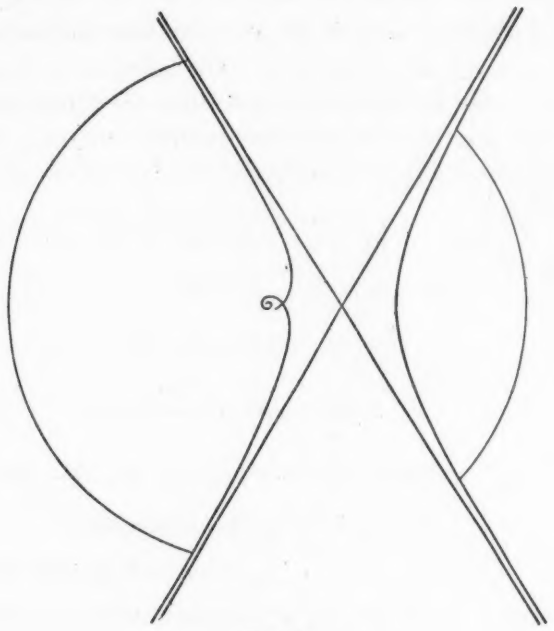

FIG. 6.

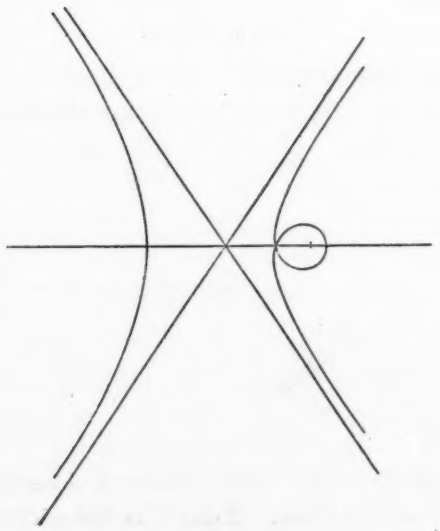

FIG. 7.

equal angles. Fig. 6 is an interesting special case, for which $a = b$, $\lambda = 0$. The equations of §6 take the form

$$2a/r = 1 - \theta^2,$$

$$\theta = \tan \frac{\psi}{2},$$

$$s = r\theta.$$

Thus the length of a symmetric arc of the curve is equal to that of the intercepted arc of a concentric circle, as shown in the figure.

The parallel with the ordinary catenary may be developed; for instance if we ask what is the least length which can rest over two pins equidistant from O, with free ends, under an attraction, it will appear that the curve must be of the form (6) and the least length is

$$2r(1+\theta)^2,$$

where r is the distance of a pin from the centre, and 2θ is the angle (supposed < 2) which the pins subtend at the centre. This we may leave as an exercise.

The equation $2a/r = 1 - \theta^2$ is obtainable from the theorem at the end of § 6; for if the curve of repulsion be a circle of radius r, then $c_1 = r = s/\theta$, whence the theorem gives

$$cr = r^2 - r^2\theta^2$$

or

$$r(1-\theta^2) = c.$$

8. Kinematics of the bicycle. The figures of this paper have been in part drawn by students of the college; and one of them, Mr. Redfield, made the interesting remark that the tractrix gives a good idea of the track of the rear wheel of a bicycle when the track of the front wheel is a circle (fig. 2). For small deviations of the rear wheel from the vertical the variation of the wheelbase is in fact very small, however great the inclination of the wheels, in the usual type of machine in which the steering post points (in the normal position) near the point of contact of the front wheel with the ground.

To examine the relative curve of the bicycle tracks, when the rear wheel is held vertical, call p and p_1 the distances from the axles to the steering head, q the distance between p and p_1. Let θ be the inclination of p to the horizontal, ϕ the angle turned through by p_1, a the radius of either wheel (the wheels are supposed equal, as in the American type of machine). Then the coordinates of the point of contact of the front wheel, referred to axes through that of the rear wheel, such that z is vertical and x the line of symmetry, are

$$x = p\cos\theta + q\sin\theta + p_1\cos\phi\cos\theta + \frac{a\sin\theta\cos\theta\sin^2\phi}{\sqrt{1-\sin^2\theta\sin^2\phi}}$$

$$y = p_1\sin\phi - \frac{a\sin\theta\sin\phi\cos\phi}{\sqrt{1-\sin^2\theta\sin^2\phi}},$$

$$z = a + p\sin\theta - q\cos\theta + p_1\sin\theta\cos\phi - a\sqrt{1-\sin^2\theta\sin^2\phi}.$$

Hence, since $z = 0$, we have

$$p_1\sin\theta\cos\phi + A = a\sqrt{\cos^2\theta + \sin^2\theta\cos^2\phi},$$

where $$A = a + p\sin\theta - q\cos\theta;$$

or $$(a^2 - p_1^2)\sin\theta\cos\phi = Ap_1 + a\sqrt{A^2 - (a^2 - p_1^2)\cos^2\theta},$$

whence, if $\tan\frac{\theta}{2} = t$, $\cos\phi$ is a rational function of t and $\sqrt{Qt}$, where Qt is a quartic in t.

Hence we can take, for the complete discussion of the relative curve, x, $y/\sin\phi$, and therefore also y^2, as elliptic functions of a parameter u.

HAVERFORD COLLEGE, MAY, 1899.

A GENERALIZATION OF FERMAT'S THEOREM.

By L. E. Dickson.

1. In a number of investigations, apparently not related to each other, there occurs the following function:

$$F(a,N) \equiv a^N - (a^{\frac{N}{p_1}} + a^{\frac{N}{p_2}} + \cdots + a^{\frac{N}{p_s}}) + (a^{\frac{N}{p_1 p_2}} + a^{\frac{N}{p_1 p_3}} + \cdots + a^{\frac{N}{p_{s-1} p_s}})$$

$$-(a^{\frac{N}{p_1 p_2 p_3}} + \cdots + a^{\frac{N}{p_{s-2} p_{s-1} p_s}}) + \cdots + (-1)^s a^{\frac{N}{p_1 p_2 \cdots p_s}},$$

a being any integer and N any positive integer whose distinct prime factors are $p_1, p_2, \ldots, p_s$. The theorem which we shall consider in the present paper is that $F(a,N)$ is divisible by N for every a and N. This theorem is a generalization of Fermat's theorem, to which it reduces when N is a prime.

In §§ 2–5 of the present paper it is explained how the function $F(a,N)$ has occurred in four distinct mathematical researches, and how from each of these points of view indirect proofs of the above mentioned generalized theorem have been obtained. In §6 two new direct proofs of this theorem are given. In §7 a third new direct proof is given, based upon a relation observed by Picquet. In §§7 and 8 some further properties of the function $F(a,N)$ are considered.

2. As far as known to the writer, the earliest occurrence of the function $F(a,N)$ is in an important paper by Schönemann.* He proved that, a being the power of a prime p^n, the number of congruences of degree N belonging to and irreducible in the Galois field of order $a = p^n$ is $\frac{1}{N} F(a,N)$. The same result was arrived at later by Pellet† and by the writer‡ independently. For $n = 1$, the result has been given by Serret§ and by Dedekind.‖ In none of these papers is there a reference to the earlier papers.

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* Grundzüge einer allgemeinen Theorie der höhern Congruenzen, deren Modul eine reelle Primzahl ist, *Crelle*, 31, 1846, pp. 269–325.

† *Comptes Rendus*, 70, 1870, pp. 328–330.

‡ *Bulletin of the American Mathematical Society*, July, 1897, pp. 381–389.

§ *Cours d'Algèbre supérieure*, 4th edition, Vol. 2, pp. 137–141.

‖ Abriss einer Theorie der höhern Congruenzen in Bezug auf einen reellen Primzahl-Modulus, *Crelle*, 54, 1857, pp. 1–26.

3. S. Kantor* has shown that the number of cyclic groups of order N in any Cremona transformation of order a in the plane is $\frac{1}{N} F(a,N)$. He first establishes the elegant recursion formula

$$a^N - a = \sum_{d>1} F(a,d), \tag{1}$$

the sum extending over all the divisors $d > 1$ of N, including N itself. He derives from (1) the value of $F(a,N)$ by a lengthy method which he carries out for certain special cases. The result may, however, be derived immediately as follows. We give to (1) the following form, due to Dedekind for $a =$ prime:

$$a^N = \sum_{d} F(a,d) \tag{1'}$$

summed for every divisor d of N including N and unity. We have taken† $F(a, 1) \equiv a$, in agreement with the fact that there are a distinct linear congruences modulo a, and p^n distinct linear congruences in the $GF[p^n]$. The determination of $F(a,N)$ as the general solution of (1′) follows at once from the following important general theorem due to Dedekind (*l. c.* p. 21 and pp. 25–26):

If, for an arbitrary integer N, we have

$$F(N) = \sum_{d} f(d),$$

the sum extending over all the divisors of N, including N and unity, and if $p_1, p_2, \ldots, p_s$ are all the distinct prime factors of N, then

$$f(N) = F(N) - \sum_{i=1}^{s} F\left(\frac{N}{p_i}\right) + \sum_{i<j}^{1\ldots s} F\left(\frac{N}{p_i p_j}\right) - \sum_{i<j<k}^{1\ldots s} F\left(\frac{N}{p_i p_j p_k}\right) + \cdots$$

A similar theorem holds if $F(N) = \Pi f(d)$.

4. Picquet‡ was led to the number $F(a,N)$ in a wholly different connection. He considers the curves of order m having at a given point on a given cubic $3m - 1$ consecutive points of intersection with the cubic. Such

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* *Annali di Matematica* (Milano), (2) Vol. 10, pp. 64–73.

† See also end of §7.

‡ Picquet, Sur une généralisation du théorème de Fermat, *Comptes Rendus*, 96, 1883, p. 1136.

a C_m, although not completely determined, must meet the cubic in a further fixed point. Proceeding with the latter point as before, we obtain a particular kind of curvilinear polygon, only the vertices of which are determined. If the first point be suitably chosen, the polygon is a closed curve; it is at the same time inscribed and circumscribed to the given cubic. The number of the summits of the polygons of n sides is

$$F[(3m-1)^2, n] - 2F(1-3m,n)$$

which is therefore a multiple of n. There are $\frac{1}{n} F(3m-1,n)$ polygons with real summits situated on the branch of the cubic which contains its inflections, and a like number of polygons on the oval branch when the latter exists and when m is even.

5. Finally, G. Koenigs was led to the numerical function $F(a,N)$ in the theory of uniform substitutions.* Let $\phi(z)$ be a uniform function and denote by $\phi_n(z)$ the operation $\phi(z)$ repeated n times. Consider the equation

$$(E_n) \qquad z - \phi_n(z) = 0.$$

If n' divides n, every root of $E_{n'}$ is a root of E_n. Those roots of E_n which verify no like equation of lower index are said to belong to the index n. If x belong to the index n, so do also the quantities

$$x,\ \phi(x),\ \phi_2(x),\ \phi_3(x),\ \ldots,\ \phi_{n-1}(x),$$

which are permuted cyclically by the uniform substitution

$$z' = \phi(z).$$

The roots belonging to the index n are therefore distributed into circular groups of n roots each. If m be the degree of the polynomials forming the numerator and denominator of $\phi(z)$, the number of roots belonging to the exponent n is proved to be $F(m,n)$ by Koenigs. Hence this number is divisible by n.

6. We have therefore a number of indirect proofs of the divisibility of $F(a,N)$ by N, the quotient expressing certain enumerations. Picquet gave a direct proof, but required the consideration of various sub-cases. A more elementary proof was given by Ed. Lucas.†

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* Sur une généralisation du théorème de Fermat, et ses rapports avec la théorie des substitutions uniformes, *Darboux Bull.* (2) VIII, p. 286.

† Lucas, Sur la généralisation du théorème de Fermat, *Comptes Rendus*, 96, 1883, p. 1300.

I have not had access to the proof given by Ed. Weyr.* I have recently found two very simple proofs of the theorem in question.

First proof. — Taking first the case in which N contains three distinct prime factors, $N \equiv r^\rho s^\sigma t^\tau$, we have

$$F(a,N) = (a^N - a^{\frac{N}{s}}) - (a^{\frac{N}{r}} - a^{\frac{N}{rs}}) - (a^{\frac{N}{t}} - a^{\frac{N}{ts}}) + (a^{\frac{N}{rt}} - a^{\frac{N}{rst}}).$$

Since each quantity in parenthesis is of the form

$$b^{s^\sigma} - b^{s^{\sigma-1}},$$

it is divisible by s^σ by Fermat's theorem. If N contained a fourth prime factor w, we should have in $F(a,N)$ four additional terms,

$$-(a^{\frac{N}{w}} - a^{\frac{N}{sw}}) + (a^{\frac{N}{rw}} - a^{\frac{N}{srw}}) + (a^{\frac{N}{tw}} - a^{\frac{N}{stw}}) - (a^{\frac{N}{rtw}} - a^{\frac{N}{srtw}}).$$

From the symmetry of $F(a,N)$, it follows that it is divisible by the other factors r^ρ, t^τ, etc.

Second proof. — The theorem follows immediately from the following congruence, which is readily verified:

$$[F(a, N)]^q - F(a, N) \equiv F(a, qN) \pmod{q}, \tag{2}$$

q being a prime number, while a and N are arbitrary integers. $F(a,qN)$ being therefore divisible by q, the theorem itself follows by simple induction.

We may put into evidence the divisibility of $F(a,N)$ by each of the prime factors of N simultaneously. For example,

$$F(a, rs) \equiv (a^r - a)^s - (a^r - a) \pmod{s}.$$

7. Theorem. — *The function $F(a,N)$ is characterized by the two properties,*

$$F(a, np^s) = F(a^{p^s}, n) - F(a^{p^{s-1}}, n) \tag{3}$$

$$F(a, p^s) = a^{p^s} - a^{p^{s-1}}, \tag{4}$$

where a is an arbitrary integer, n any integer not divisible by the prime number p.

We may verify by induction this theorem, stated without proof by Pic-

* Ueber einen Zahlen-theoretischen Satz, *Casopis, Zeitschrift zur Pflege der Mathematik und Physik*, XI, p. 39, 1882.

quet. Taking $n=q^r$, $q=$ prime, we have as the first step in the induction, on applying (3) and (4),

$$F(a, q^r p^s) = (a^{p^s})^{q^r} - (a^{p^s})^{q^{r-1}} - (a^{p^{s-1}})^{q^r} + (a^{p^{s-1}})^{q^{r-1}},$$

giving therefore the function $F(a,N)$ of §1 for the case $N=q^r p^s$. To give the proof of the general step in the induction, we assume, for a given integer n having the prime factors $p_1, \ldots p_r$, and an arbitrary integer b, that the function $F(b, n)$ defined by (3) and (4) is precisely the function $F(b, n)$ of §1. Taking $b=a^{p^s}$ and $a^{p^{s-1}}$ in turn, we have by assumption the identities,

$$F(a^{p^s}, n) = a^{np^s} - (a^{\frac{n}{p_1}\cdot p^s} + \cdots + a^{\frac{n}{p_r}\cdot p^s}) + a^{\frac{n}{p_1 p_2}\cdot p^s} + \cdots\cdots$$

$$F(a^{p^{s-1}}, n) = a^{np^{s-1}} - (a^{\frac{n}{p_1}\cdot p^{s-1}} + \cdots + a^{\frac{n}{p_r}\cdot p^{s-1}}) + a^{\frac{n}{p_1 p_2}\cdot p^{s-1}} + \cdots\cdots$$

Hence if p be a prime number not contained in n, we have by applying (3),

$$F(a, np^s) = a^{np^s} - (a^{np^{s-1}} + a^{\frac{np^s}{p_1}} + \cdots + a^{\frac{np^s}{p_r}})$$
$$+ (a^{\frac{np^s}{p\,p_1}} + \cdots + a^{\frac{np^s}{p\,p_r}} + a^{\frac{np^s}{p_1 p_2}} + \cdots) - \cdots\cdots,$$

which is precisely the function $F(aN)$ of §1, for $N=np^s$.

As a corollary we may derive a very simple proof (not observed by Picquet however) of the divisibility of $F(a,N)$ by N. The function $F(a, p^s)$ defined by (4) is divisible by p^s by Fermat's theorem. If then we assume that $F(a,n)$ is divisible by n for a arbitrary, it follows from (3) that $F(a, np^s)$ is divisible by n, p being a prime not contained in n. Our corollary thus follows by induction. This proof is closely related to the first proof given in §6.

We may generalize the relation (3) as follows. If m have the distinct prime factors $r, s, t, \ldots, w$ and if n be relatively prime to m, then

$$F(a,nm) = F(a^m,n) - F(a^{\frac{m}{r}},n) - F(a^{\frac{m}{s}},n) - \cdots\cdots - F(a^{\frac{m}{w}},n)$$
$$+ F(a^{\frac{m}{rs}},n) + \cdots\cdots + F(a^{\frac{m}{vw}},n) - \cdots\cdots \pm F(a^{\frac{m}{rs\cdots w}},n).$$

This formula reduces to that of §1 for $n=1$, $m=N$, provided we take $F(b, 1) \equiv b$.

8. An interesting number-theoretic property of the function $F(a,N)$ is given by the following theorem, which I believe to be new:

If $\phi(d)$ *denotes the number of positive integers prime to* d *and not exceeding* d, *then, for arbitrary integers* a *and* N *greater than unity,*

$$\sum_d \phi(d) = F(a,N)$$

the sum extending over all proper divisors of $a^N - 1$.

By a *proper* divisor of a^N-1 we understand one which does not divide a^m-1 if $m<N$. Thus, the proper divisors of 6^4-1 are $3\cdot7\cdot37$, $5\cdot37$, $7\cdot37$, and 37; the sum of their ϕ's equals $35\cdot36=6^4-6^2=F(6,4)$. Again, the proper divisors of 6^6-1 are $5\cdot7\cdot31\cdot43$, $5\cdot31\cdot43$, $7\cdot31\cdot43$, $5\cdot7\cdot43$, $31\cdot43$, $7\cdot43$, $5\cdot7\cdot31$, $7\cdot31$, $5\cdot31$, and 31, the sum of whose ϕ's equals $30\cdot1547=6^6-6^3-6+6=F(6,6)$.

The proof of the theorem follows immediately from the above general theorem of Dedekind, if we note that

$$\sum_n \sum_d \phi(d) = a^N - 1,$$

n running through all the divisors of N including N and unity. Further, $F(a,N)$ is unchanged if we replace every term $a^{\frac{N}{K}}$ by $a^{\frac{N}{K}}-1$, the added terms being given by the expansion of $-(1-1)^s$.

UNIVERSITY OF CALIFORNIA, APRIL, 1899.

EXAMPLES IN THE THEORY OF FUNCTIONS.*

By Maxime Bôcher.

1. $f_1(z)$ and $f_2(z)$ are analytic functions. Let $F(z) = f_1(z) \cdot f_2(z)$. What can we say about $F(z)$ at a point a if $f_1(z)$ and $f_2(z)$ both have poles at a? If $f_1(z)$ and $f_2(z)$ both have essentially singular points at a?

2. If a polynomial in x and y with real coefficients satisfies Laplace's equation, prove that it is the real part of a polynomial in $x + yi$.

Prove that the necessary and sufficient condition that a homogeneous polynomial of the n^{th} degree in x and y satisfy Laplace's equation, is that the equation formed by setting the polynomial equal to zero represent n real straight lines making angles π/n with one another.

3.† A function $u(x, y,)$ is harmonic‡ in a certain region of the plane. Prove that it can be developed about any point (x_0, y_0) of this region in a series of the form:

$$u(x, y) = \Phi_0 + \Phi_1 + \Phi_2 + \cdots$$

where each term is itself harmonic and where Φ_n is a homogeneous integral rational function of the n^{th} degree of $(x - x_0)$ and $(y - y_0)$.

State this proposition using polar coordinates (r, ϕ), the point (x_0, y_0)

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* In teaching any branch of mathematics it is essential not merely to illustrate the theory by examples, but to give the student exercises in considerable number to work out for himself. Moreover, these exercises should not all be capable of solution by *direct* application of the particular principle last considered; the student should learn from them to do something more than merely "follow copy." It is often difficult to find a sufficient variety of such examples, especially in the more recently developed portions of mathematics, and it requires no little labor on the part of the instructor to invent them.

It is with the hope of assisting teachers who experience this difficulty that the following collection of problems has been prepared. If it prove useful we shall hope to publish, at irregular intervals, similar collections in other branches of mathematics; and we shall welcome contributions of single problems or sets of problems for this purpose. We have not intentionally included examples which are contained in any of the ordinary text-books on the subject, but aside from this no restriction has been laid on their source, some being taken with more or less modification from various books and memoirs, while others are entirely original.

[Ed.]

† It is suggested that the propositions in 3 and 4 be established by deducing them from similar theorems concerning analytic functions of a complex variable.

‡ A function is said to be harmonic in a region if it is single-valued and continuous throughout this region, has continuous first and second partial derivatives at every point of the region, and at every point of the region satisfies Laplace's equation.

being taken as origin. Determine the coefficients in the series thus obtained in the form of line integrals.

4.* How must the theorem of the last problem be modified if (x_0, y_0) is an isolated singular point *i. e.* if $u(x,y)$ is harmonic everywhere in the neighborhood of (x_0, y_0), but not at (x_0, y_0) itself?

Hence distinguish between the various kinds of isolated singular points of harmonic functions, and discuss the behavior of the function in the neighborhood of each kind of singular point.

In particular prove that if u becomes infinite for every method of approaching (x_0, y_0), it is of the form $C \log \sqrt{(x-x_0)^2 + (y-y_0)^2} + u_1(x,y)$ where u_1 is harmonic at (x_0, y_0) as well as elsewhere.

5. Prove geometrically† that if we have two circles C_1 and C_2 in the complex plane, and invert the plane first with regard to C_1 and then with regard to C_2, the total effect will be the same as that of a certain linear transformation $w = \dfrac{\alpha z + \beta}{\gamma z + \delta}$.

Prove that reflexion of the complex sphere in the plane of any great circle C corresponds to inversion of the complex plane with regard to the stereographic projection of C.

Prove that rotation (through any angle) of the complex sphere about a diameter, corresponds to a certain linear transformation.

(Suggestion: show that the rotation is equivalent to the succession of two reflexions in diametral planes.)

6. The linear transformation $z' = \dfrac{\alpha z + \beta}{\gamma z + \delta}$ will in general leave two points of the plane fixed. Find the necessary and sufficient condition that infinitesimal portions of the plane surrounding those points should be:

(α) turned but not stretched,
(β) stretched but not turned,

by the transformation.

Obtain from the above a necessary condition that the linear transformation should represent a rotation of the sphere.

7. Prove that if it is possible to define sin z and cos z as analytic func-

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* See foot-note marked † on previous page.
† For an analytical proof see *Forsyth*, p. 523.

tions of z which reduce when z is real to the ordinary sine and cosine, the following formulae will hold:

$$\sin^2 z + \cos^2 z = 1,$$
$$\sin z_1 \cos z_2 + \cos z_1 \sin z_2 = \sin (z_1 + z_2).$$

Avoid the use of any explicit formulae for $\sin z$ and $\cos z$ in this proof.

8. Given two points a and b in the complex plane, a curve C connecting them, and a function $f(t)$ continuous for all values of t which lie on this curve. Prove that

$$F(z) = \int_a^b \frac{f(t)}{t - z} dt$$

(the integral being taken along C) is a monogenic function of z at all points of the plane which do not lie on C. What kind of a point does $F(z)$ have at infinity?

9. The function $f(z)$ is analytic throughout a region S; a is a point of this region; and b is any point of the plane. Prove that in the neighborhood of a $f(z)$ can be developed in the form $\sum_{n=0}^{n=\infty} C_n \left(\frac{z-a}{z-b}\right)^n$. What can we say about the region throughout which this series converges and represents $f(z)$?

10. Given the functions $u_0(z)$, $u_1(z)$, $u_2(z)$, . . . analytic throughout a certain finite region S of the z-plane. Given further that the series:

$$u_0(z) + u_1(z) + u_2(z) + \cdots$$

converges uniformly throughout S. Denote the function defined by this series by $f(z)$, and let a be a root of the k^{th} order of $f(z)$. Prove that if a circle be drawn about a as centre and of radius so small that it lies wholly within S and does not include or pass through any root of $f(z)$ except a, then the function represented by the sum of the first n terms of the above series will have exactly k roots in this circle for sufficiently large values of n.

11. In the ring-shaped region bounded by two circles with centres at the origin and radii r_1 and r_2 the function $f(z)$ is single-valued and analytic except for a finite number of poles. Moreover this function joins on continuously to continuous boundary values at all points of both circles, and these boundary values are the same at the two points where any radius cuts the two circles. Prove that this function admits of analytic extension all over the plane.

Determine the function completely in case there are no poles in the original region; and in other cases obtain such properties of the function as you can.

12. $f(z)$ is analytic on a piece of a four-sheeted Riemann's Surface bounded by a large circle and having two branch points of the first order at a_1 connecting the first and second and the third and fourth sheets respectively, and a branch point at a_2 connecting the second and third sheets. Construct a piece of a Riemann's Surface of the same size and shape upon which $\sqrt[3]{(z-a_1)(z-a_2)}\cdot f(z)$ is analytic.

13. Construct the Riemann's Surface for the following functions:

$$(a) \qquad w=\sqrt{\sin z},$$
$$(b) \qquad w=\sqrt{\tan z},$$
$$(c) \qquad w=\sqrt{e^z}.$$

Describe the singularities of each of these functions.

14. Study by the method of conformal representation the function:

$$w=\int_0^z (1-z^2)^{-\frac{1}{2}}dz$$

and its inverse.

CAMBRIDGE, MASS., JUNE, 1899.

THE IRREDUCIBLE CASE OF THE CUBIC EQUATION.

By Alston Hamilton.

The most general form of the equation of the third degree is

$$x^3 + Ax^2 + Bx + C = 0.$$

In this, place $$x = y - \frac{A}{3}.$$

This gives the following equation involving y:

$$y^3 + py + q = 0,$$

where $$p = \frac{3B - A^2}{3},$$

and $$q = \frac{2A^3}{27} - \frac{AB}{3} + C.$$

Cardan's Rule applies to the equation $y^3 + py + q = 0$, and the result is satisfactorily obtainable when

$$\frac{q^2}{4} + \frac{p^3}{27} \geqq 0.$$

This inequality is satisfied whenever p is a positive quantity, and also when p is both negative and numerically less than or equal to $\frac{3}{2}\sqrt[3]{2q^2}$.

If Cardan's Rule "fails" we have the "Irreducible Case," so called because it involves roots of imaginary quantities the extraction of which is again dependent upon cubic equations.

For the sake of greater clearness we replace p, which in irreducible cases has been shown to be essentially negative, by $-p'$.

Then $$y^3 - p'y + q = 0. \qquad (1)$$

Now place $$y = z\sqrt{p'}.$$

Then, by substitution in equation (1)

$$z^3 - z + \frac{q}{p'\sqrt{p'}} = 0.$$

Place $$\frac{q}{p'\sqrt{p'}} = q';$$

then $$z^3 - z + q' = 0.$$

Determine the function completely in case there are no poles in the original region; and in other cases obtain such properties of the function as you can.

12. $f(z)$ is analytic on a piece of a four-sheeted Riemann's Surface bounded by a large circle and having two branch points of the first order at a_1 connecting the first and second and the third and fourth sheets respectively, and a branch point at a_2 connecting the second and third sheets. Construct a piece of a Riemann's Surface of the same size and shape upon which $\sqrt[3]{(z-a_1)(z-a_2)}\cdot f(z)$ is analytic.

13. Construct the Riemann's Surface for the following functions:

$$(a) \qquad w=\sqrt{\sin z},$$
$$(b) \qquad w=\sqrt{\tan z},$$
$$(c) \qquad w=\sqrt{e^z}.$$

Describe the singularities of each of these functions.

14. Study by the method of conformal representation the function:

$$w=\int_0^z (1-z^2)^{-\frac{1}{2}}dz$$

and its inverse.

CAMBRIDGE, MASS., JUNE, 1899.

THE IRREDUCIBLE CASE OF THE CUBIC EQUATION.

By Alston Hamilton.

The most general form of the equation of the third degree is

$$x^3 + Ax^2 + Bx + C = 0.$$

In this, place $$x = y - \frac{A}{3}.$$

This gives the following equation involving y:

$$y^3 + py + q = 0,$$

where $$p = \frac{3B - A^2}{3},$$

and $$q = \frac{2A^3}{27} - \frac{AB}{3} + C.$$

Cardan's Rule applies to the equation $y^3 + py + q = 0$, and the result is satisfactorily obtainable when

$$\frac{q^2}{4} + \frac{p^3}{27} \geqq 0.$$

This inequality is satisfied whenever p is a positive quantity, and also when p is both negative and numerically less than or equal to $\frac{3}{2}\sqrt[3]{2q^2}$.

If Cardan's Rule "fails" we have the "Irreducible Case," so called because it involves roots of imaginary quantities the extraction of which is again dependent upon cubic equations.

For the sake of greater clearness we replace p, which in irreducible cases has been shown to be essentially negative, by $-p'$.

Then $$y^3 - p'y + q = 0. \qquad (1)$$

Now place $$y = z\sqrt{p'}.$$

Then, by substitution in equation (1)

$$z^3 - z + \frac{q}{p'\sqrt{p'}} = 0.$$

Place $$\frac{q}{p'\sqrt{p'}} = q';$$

then $$z^3 - z + q' = 0.$$

Enter the table on page 45 with the numerical value of q' as an argument and by interpolation find z. Take the sign of z the opposite of that of q'. Then

$$x = y - \frac{A}{3} = z\sqrt{p'} - \frac{A}{3}.$$

This is the outline of the *use* of the method. An example follows, and then follows the theory on which the preceding method is based.

EXAMPLE.

$$x^3 + 3x^2 - 5x + 1 = 0.$$

Here $$p = \frac{3B - A^2}{3} = -8,$$

and $$q = \frac{2A^3}{27} - \frac{AB}{3} + C = +8;$$

$\frac{q^2}{4} + \frac{p^3}{27} = -2\frac{26}{27}$, which < 0, and Cardan's Rule "fails."

Place $$y = x + \frac{A}{3} = x + 1,$$

then $$y^3 - 8y + 8 = 0.$$

Place $$y = z\sqrt{p'} = z\sqrt{8},$$

then $$z^3 - z + \frac{1}{4}\sqrt{2} = 0,$$

or $$z^3 - z + .353553 = 0.$$

Now refer to the table, and we find that

for $z = 1.144$ . . . . . $q' = .353194$,
for $z = 1.145$ . . . . . $q' = .356124$.

Finding by interpolation from this the value of z corresponding to $q' = .353553$

we have $$z = -1.1441225,$$

and $$y = z\sqrt{8} = -3.236067,$$

hence $$x = y - 1 = -4.236067.$$

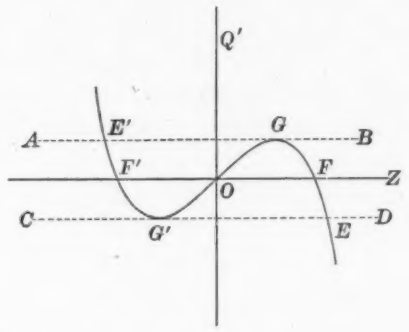
This is the method employed. The *reasons* follow:

The equation being in the form

$$z^3 - z + q' = 0,$$

we may by assigning "consecutive" values to z obtain corresponding values

for the arbitrary constant q', and plotting the result we obtain the curve whose ordinate is q'



Now examine the curve for maximum and minimum points.

From $$z^3 - z + q' = 0$$

we have by differentiation $$-\frac{dq'}{dz} = 3z^2 - 1;$$

and for a maximum or minimum point,

$$\frac{dq'}{dz} = 0 = 3z^2 - 1;$$

whence $z = \pm \frac{1}{3}\sqrt{3}$ and the corresponding values q' are $q' = \pm \frac{2}{9}\sqrt{3}$.

Draw the lines AB and CD parallel to the axis of z. Their equations are, respectively

$$q' = +\frac{2}{9}\sqrt{3},$$

$$q' = -\frac{2}{9}\sqrt{3}.$$

Now revert to the condition of applicability of Cardan's Rule

$$\frac{q'^2}{4} - \frac{1}{27} \geqq 0.$$

This gives q' a value greater numerically than or equal numerically to $\frac{2}{9}\sqrt{3}$. Then, in our curve, the division between solubility and insolubility by Cardan's Rule is along the lines AB and CD.

It is interesting to note that for values of q' numerically greater than $\frac{2}{9}\sqrt{3}$, there is only one real value of z, while for values of q' numerically less than $\frac{2}{9}\sqrt{3}$, there are three real values of z (this being all in reference to the curve), and at the same time to remember that Cardan's Rule is applied in practice only to equations having one real root and two imaginary roots, or two equal roots.

Hence we confine ourselves to the portion of the curve between the lines AB and CD, the "Irreducible Case." Observing the form of the curve, we see that by selecting any one of the six arcs EF, FG, GO, $E'F'$, $F'G'$, $G'O$, z may be given values in the selected segment, at small regular intervals, from which corresponding values of q' may be obtained, since $z^3 - z = -q'$. If a sufficient number of such values be obtained a table will result, from which, either z or q' being given, the other may be found by interpolation to any desired degree of accuracy, the accuracy depending upon the magnitude of the interval between successive values of z.

I have selected the arc $E'F'$ because in that portion of the curve q' changes more rapidly than z. This is an advantage, q' being the argument with which the table is generally entered and z the value to be found. The sign of q' does not affect the validity of the table if we remember that the sign of z is always to be taken opposite to that of q', since reversing the signs of z and q' does not affect the equation $z^3 - z + q' = 0$. Hence, whatever the sign of q', provided its value lies between 0 and $\pm\frac{2}{9}\sqrt{3}$, the table above indicated gives one value of z corresponding to the value of q'.

Having found from the table a value of z lying on $E'F'$, taking care to prefix the proper sign, the two other values may be found from a quadratic equation, or similar tables may be constructed for the segments OG and GF.

The table attached includes values of z from 1.000 to 1.160, and gives quite accurate results. It will enable one to solve any cubic equation with real coefficients that is "irreducible."

TABLE FOR THE SOLUTION OF IRREDUCIBLE CASES OF THE CUBIC EQUATION.

z	0	1	2	3	4	5	6	7	8	9
1.00	.000000	.002003	.004012	.006027	.008048	.010075	.012108	.014147	.016193	.018244
1.01	.020301	.022364	.024434	.026509	.028591	.030678	.032772	.034872	.036978	.039090
1.02	.041208	.043332	.045463	.047599	.049742	.051891	.054046	.056207	.058374	.060547
1.03	.062727	.064913	.067105	.069303	.071507	.073718	.075935	.078158	.080387	.082622
1.04	.084864	.087112	.089366	.091627	.093893	.096166	.098445	.100731	.103023	.105321
1.05	.107625	.109936	.112253	.114576	.116905	.119241	.121584	.123932	.126287	.128648
1.06	.131016	.133390	.135770	.138157	.140550	.142950	.145355	.147768	.150186	.152612
1.07	.155043	.157481	.159925	.162376	.164833	.167297	.169767	.172244	.174727	.177216
1.08	.179712	.182214	.184723	.187239	.189761	.192289	.194824	.197366	.199913	.202468
1.09	.205029	.207597	.210171	.212751	.215339	.217932	.220533	.223140	.225753	.228373
1.10	.231000	.233633	.236273	.238920	.241572	.244233	.246899	.249572	.252252	.254938
1.11	.257631	.260331	.263037	.265750	.268470	.271196	.273929	.276669	.279415	.282168
1.12	.284928	287695	.290468	.293248	.296035	.298828	.301628	.304435	.307249	.310070
1.13	.312897	.315731	.318572	.321420	.324274	.327135	.330003	.332878	.335760	.338649
1.14	.341544	.344446	.347355	.350271	.353194	.356124	.359060	.362004	.364954	.367911
1.15	.370875	.373846	.376824	.379809	.382800	.385799	.388804	.391817	.394836	.397863
1.16	.400896									

NOTE.— The first column contains values of z. The numbers 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 at the heads of the other columns belong to z, carrying its value to thousandths. All the columns but the first contain values of q'. The sign of z is always the opposite of that of q'.

MANILA, PHILIPPINE ISLANDS, JUNE, 1899.

A NOTE ON CRITICAL POINTS.

By F. H. Safford.

In the Annals of Mathematics, February, 1898, Dr. C. L. Bouton mentions, on page 26, the following function:

$$w=\frac{\sum_{k=0}^{k=p}\frac{(-1)^k z^{n-km}}{(p-k)!\,k!\,[n-km]\,[n-(k+1)m]\ldots[n-(k+p)m]}}{\sum_{k=0}^{k=p}\frac{(-1)^k z^{(p-k)m}}{(p-k)!\,k!\,[n-(p-k)m]\,[n-(p-k+1)m]\ldots[n-(2p-k)m]}}$$

where $1\leqq m\leqq\frac{n-1}{p}$, $m\neq\frac{n}{p+k}$ for $k=1,2\ldots p$; and $n\geqq 2p+1$; and says that it seems probable that the critical points are given by the equation:

$$z^{n-pm-1}(z^m-1)^{2p}=0.$$

To prove this surmise correct, multiply the numerator and denominator of w by the expression: $p![n(n-m)\ldots\ldots(n-pm)]$ and write the denominator in reverse order.

Letting $z^m=x$, $\frac{n}{m}=s=-\alpha$, $p=-\beta$, $p-s+1=\gamma$, we get for $(-1)^p w$ the value

$$\frac{x^s+\frac{\alpha.\beta}{1.\gamma}x^{s-1}+\frac{\alpha(\alpha+1).\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^{s-2}\ldots\frac{\alpha(\alpha+1)\cdots(\alpha+p-1).\beta\ldots(\beta+p-1)}{1.2\ldots p.\gamma\ldots(\gamma+p-1)}x^{s-p}}{1+\frac{\alpha.\beta}{1.\gamma}x+\frac{\alpha(\alpha+1).\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2\ldots\frac{\alpha(\alpha+1)\ldots(\alpha+p-1).\beta\ldots(\beta+p-1)}{1.2.\ldots p.\gamma\ldots(\gamma+p-1)}x^p}$$

The denominator of this fraction is:

$$y_1=F(a,\beta,\gamma,x)=F(-s,-p,p-s+1,x),$$

and its numerator is:

$$y_2=x^s F\left(a,\beta,\gamma,\frac{1}{x}\right)=x^s F\left(-s,-p,p-s+1,\frac{1}{x}\right),$$
$$=x^{-a}F\left(a,a-\gamma+1,a-\beta+1,\frac{1}{x}\right).$$

For the critical points of w, as usual,

$$\frac{dw}{dz}=mz^{m-1}\frac{dw}{dx}=mz^{m-1}(-1)^p\left[y_1\frac{dy_2}{dx}-y_2\frac{dy_1}{dx}\right]/y_1^2=0.$$

From Forsyth's Differential Equations pp. 192, 193, the values of y_1 and y_2 are merely his solutions I and IX, respectively, of the differential equation satisfied by $F(\alpha, \beta, \gamma, x)$. But from p. 201, art. 128

$$y_1\frac{dy_2}{dx}-y_2\frac{dy_1}{dx}=Cx^{-\gamma}(1-x)^{\gamma-\alpha-\beta-1}.$$

Replacing x, α, β, γ, by their values in terms of z, m, n, p, and bringing down the factor z^{m-1}, we get, finally, omitting constant factors,

$$z^{n-mp-1}(z^m-1)^{2p}=0.$$

CAMBRIDGE, MASS., APRIL, 1899.

NOTE ON THE FUNCTION SATISFYING THE FUNCTIONAL RELATION $\phi(u)\cdot\phi(v)=\phi(u+v)$.*

BY E. B. WILSON.

The exponential function a^x,

1) is defined for all values of the argument x,

2) is single-valued,

3) satisfies the functional relation $\phi(u)\cdot\phi(v)=\phi(u+v)$.

Conversely any function which satisfies conditions 1, 2, 3 and is *continuous*, is the exponential function $y=[\phi(1)]^x$.†

The fundamental property on which the proof rests is that if x_0 be any value whatsoever and x any rational number, $\phi(x\cdot x_0)=[\phi(x_0)]^x$. This shows, to speak geometrically, that values of the variable commensurable with any particular value x_0, give points lying on the curve $y=[\phi(x_0)]^x$. These points do not cover the curve completely, but they are everywhere dense. If the condition of continuity, which up to this point of the proof has not been used, be now added, the function is completely determined as $[\phi(1)]^x$.

Whether or not a *discontinuous* function may satisfy the relation $\phi(u)\cdot\phi(v)=\phi(u+v)$ has, so far as the author of this note is aware, never been decided. We shall however prove the

THEOREM: *If a function exists, which is defined for all values of the argument, is single-valued, satisfies the functional relation* $\phi(u)\cdot\phi(v)=\phi(u+v)$,

---

* This paper was presented to the American Mathematical Society at its meeting of April 29, 1899.

† For proof *cf.* Tannery: *Functions d'une Variable* §80, p. 120.

and has one discontinuity, then, in the neighborhood of any value x_0 of the argument, the function takes on values arbitrarily near to any preassigned positive value y_0.

Proof: Form the auxiliary function

$$\psi(x) = \frac{\phi(x)}{[\phi(1)]^x} \text{ *}.$$

If x is rational $\phi(x) = \phi(x \cdot 1) = [\phi(1)]^x$ as above stated. Therefore $\psi(x) = 1$ when x is rational. Moreover, $\psi(u) \cdot \psi(v) = \psi(u + v)$. Hence if a and b be any two *rational* numbers $\psi(ax + b) = \psi(ax) \cdot \psi(b) = [\psi(x)]^a \cdot 1 = [\psi(x)]^a$.

Let x' be a value of x for which $\phi(x)$ is discontinuous. $\psi(x)$, which is the ratio of $\phi(x)$ to a continuous function $[\phi(1)]^x$ must also be discontinuous at x'. Hence in the neighborhood of x' there must be a value x'' of x such that $\psi(x'') \neq 1$. $x'' \neq 0$, for $\psi(0) = 1$.

Since $\psi(x'') \neq 1$,† obviously

1) a rational number a can be so chosen that

$$| [\psi(x'')]^a - c | < \gamma,$$

c, γ being any preassigned arbitrary positive numbers; and

2) a second rational number b can be so chosen that

$$| a x'' + b - x_0 | < \delta,$$

x_0 being any arbitrary preassigned number, and δ any arbitrary positive number. It has, however, been shown already that $[\psi(x'')]^a = \psi(ax'' + b)$. Hence $| \psi(ax'' + b) - c | < \gamma$, and $| ax'' + b - x_0 | < \delta$.

To put these inequalities in words:—

In the neighborhood of any preassigned value x_0 of x, the function $\psi(x)$ will take on values arbitrarily near to any preassigned positive value. Or geometrically: Every two-dimensional region of the upper half plane contains points of the locus $y = \psi(x)$.

The equation $\phi(x) = \psi(x) \cdot [\phi(1)]^x$ connects $\psi(x)$ and $\phi(x)$. $[\phi(1)]^x$ is a continuous function which never takes on the value zero. Hence $\psi(x) \cdot [\phi(1)]^x$ will give, just as $\psi(x)$ did, points in every two-dimensional region of the upper half plane, *i. e.* $\psi(x) \cdot [\phi(1)]^x$ takes on values arbitrarily near to any preassigned positive value.

CAMBRIDGE, MASS., APRIL, 1899.

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* $\phi(1) \neq 0$, as otherwise $\phi(1) \cdot \phi(-1) = \phi(0) = 0$; but $\phi(0) = 1$ *cf.* Tannery.

† $\psi(x'') > 0$, for $\phi(x) > 0$ *cf.* Tannery.

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